February 23rd, 2016 **Name (Please Print)** ________________________
Special Topics in Probability - Midterm - Semester II

Instructions: Maximum time is 3 hours. If you are using a result shown in class then please state it precisely.

1. Let $\alpha > 0$. Consider the graph $\mathbb{Z}_+$ with weights $\mu^{\alpha}_{n,n+1} = \alpha^n$.
   (a) (10 points) Show the graph $(\mathbb{Z}_+, \mu^{\alpha})$ has controlled weights. Does it have bounded weights ?
   (b) (20 points)Show that the graph is recurrent if and only if $\alpha \leq 1$.

2. Let X_n be a random walk on $\mathbb{Z}^3$ with natural weights. Let T_0 be the hitting time of 0. Let $n \geq 1$, $A = \mathbb{Z}^3 \setminus \{0\}$. Let $h_n, h : \mathbb{Z}^3 \to [0, 1]$ be given by
$$h_n(x) = \mathbb{P}^x(T_0 \geq n) = \mathbb{P}^x(X_k \in A, 1 \leq k \leq n)$$
and
$$h(x) = \mathbb{P}^x(T_0 = \infty) = \mathbb{P}^x(X_n \in A, \text{ for all } n \geq 0).$$
   (a) (10 points) Show that $h_n = Q^n 1_A$ and $h = Qh$, where Q is the restriction of P onto A.
   (b) (5 points) Suppose $\alpha = \sup_{x\in A} h(x)$, show that $0 < \alpha \leq 1$ and $h \leq \alpha 1_A$
   (c) (5 points) Using (i) and (ii), conclude that $h \leq \alpha h_n$
   (d) (5 points) Conclude that $\max_{x\in\partial A} h(x) \neq \sup_{x\in\bar{A}} h(x)$.

3. Consider Γ to be the join of two copies of $\mathbb{Z}^3$ at their origins. Write $\mathbb{Z}^3_{(i)}, i = 1, 2$ the two copies, and 0_i for their origins. Let
$$F = \{X \text{ is ultimately in } Z^3_{(1)}\}$$
and let $h(x) = \mathbb{P}^x(F)$.
   (a) (10 points) Show that h is harmonic,
   (b) (5 points) Show that $h(x) \geq \mathbb{P}^x(X \text{ never hits } 0_1)$ for $x \in \mathbb{Z}^3_{(1)}$.
   (c) (5 points) Show that $h(x) \leq \mathbb{P}^x(X \text{ hits } 0_2)$ for $x \in \mathbb{Z}^3_{(2)}$.
   (d) (5 points) State the Liouville Property and decide whether Γ satisfies it.

4. Let $(\Gamma = (V, E), \mu)$ be a locally finite, connected, infinite vertex, weighted graph. Let $\Omega = V^{\mathbb{Z}_+}$. For any $n \geq 0$, let $X_n : \Omega \to V$ be given by $X_n(\omega) = \omega_n$,
$$\mathcal{F}_n = \sigma\{X_k : 0 \leq k \leq n\}, \mathcal{G}_n = \sigma\{X_k : 0 \leq k \geq n\}, \text{ and } \mathcal{F} = \mathcal{G}_0 = \sigma\{X_n : n \geq 0\}.$$
   (a) (5 points) Define the invariant σ-field, $\mathcal{I}$.
   (b) (5 points) Define the tail σ-field, $\mathcal{T}$.
   (c) (10 points) Show that $\mathcal{I} \subset \mathcal{T}$.
   (d) (5 points) Given an example where $\mathcal{I} \subsetneq \mathcal{T}$.